Reply to arXiv:0711.4930[hep-th] by Ito and Seiler

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Abstract

In a recent note (arXiv:0711.4930[hep-th]) Ito and Seiler claim that there is a 'missing link' in the derivation in arXiv:0707.2179[hep-th] by the present author; namely, that no proof of a certain inequality used there is given at weak coupling. Here it is pointed out that in fact no such missing link is present. The argument in 0707.2179 is, among other things, specifically constructed so that the inequality in question is invoked *only* at strong coupling, where it is easily proven. Underlying the mangling of the argument in 0707.2179 by Ito and Seiler are their incorrect statements concerning the dependence of the potential-moving decimation procedures used in 0707.2179 on space-time dimensionality and other decimation parameters.

1 Introduction and conclusion

In a recent paper by this author [1] a derivation is given of the existence in $SU(2)$ gauge theory of a confining phase for all values of the coupling. In [3] K.R. Ito and E. Seiler claim to have found 'missing links' in this derivation.

Missing links in a complete argument can be trivially created by simply omitting parts of the argument. This is all that Ito and Seiler do in their note. Specifically, they focus on the proof of inequality (5.15) in [1], restated as (3.5) in [3]. This inequality is easily proven in the strong coupling regime, i..e the region of convergence of the strong coupling expansion. However, there is no proof of it in the weak coupling regime. This is their 'missing link'.

In fact, no such gap in the derivation is present: the inequality in question is invoked *only* at strong coupling. Indeed, the development leading up to the appearance of the inequality is laid out in explicit detail in the first part of section 5 (p. 24 - 26) in [1]. It is specifically constructed so that, by first allowing for a sufficently large number of RG decimations, the RG flow enters the regime of the convergence of the strong coupling expansion. It is only at this point that the further development of the argument leads to invoking inequality (5.15). In other words, by construction, inequality (5.15) does not come up at weak coupling (large β).

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Though all this should be clear to anyone with even a cursory familiarity with the argument in [1], we go through the obligatory chore of spelling this out in some more detail in section 2 below. The reader familiar with the argument in [1] can skip this and proceed to the next section.

Underlying the misrepresentation of the argument in [1] by Ito and Seiler are various incorrect assertions in [3] concerning the workings of the potential-moving decimations employed in [1]. This is discussed in section 3.

2 The procedure followed in [1]

The presentation of the various derivation steps, as well as their motivation, is quite explicit in [1], and the interested reader is referred to it for full details. A brief outline of the basic steps is given in [2]. Still, in view of the mangling by Ito and Seiler of the actual procedure laid out in [1], it will be helpful to briefly reiterate some relevant parts of it here.

Starting from a given lattice action A_p, such as the Wilson action, on a lattice Λ, we work in terms of the character expansion $\exp A_p(U) = F_0 \left[1 + \sum_{j\neq 0} d_j \, c_j(\beta) \, \chi_j(U) \right]$. The partition function on Λ is then defined by

$$Z_\Lambda(\beta) = \int dU_\Lambda \, \prod_p \, \Big[1 + \sum_{j\neq 0} d_j \, c_j(\beta) \, \chi_j(U) \Big] \equiv Z_\Lambda\Big(\{c_j(\beta)\} \Big) \, . \qquad (2.1)$$

Decimations are next introduced whereby the lattice spacing a is changed by a scale factor b in successive steps: $a \to ba \to b^2 a \to \cdots \to b^n a$, generating the corresponding decimated lattices: $\Lambda \to \Lambda^{(1)} \to \Lambda^{(2)} \to \cdots \to \Lambda^{(n)}$. Decimation operations of the 'potential-moving type' are employed, which preserve the form (2.1). Such a decimation operation can be summarized as a set of rules for the computation of the coefficients of the character expansion at the $(m+1)$-th step in terms of those of the m-th step:

$$\begin{aligned} F_0(m) &= F_0(\zeta, b, \{c_i(m-1)\}) &(2.2)\\ c_j(m) &= c_j(\zeta, r, b, \{c_i(m-1)\}) \, . &(2.3) \end{aligned}$$

The explicit forms of (2.2) - (2.3) are given in [1], section 2. Here we only note that they involve parameters ζ, r which control the amount by which the plaquette interactions remaining after a decimation step are 'renormalized' to compensate for the ones that were removed. Correspondingly, the partition function undergoes the transformation

$$Z_{\Lambda^{(m-1)}}\Big(\{c_j(m-1)\} \Big) \to F_0(m)^{|\Lambda^{(m)}|} \, Z_{\Lambda^{(m)}}\Big(\{c_j(m)\} \Big) \, . \qquad (2.4)$$

There is a bulk free energy contribution from the blocking $b^{m-1}a \to b^m a$, whereas $Z_{\Lambda^{(m)}}\Big(\{c_j(m)\} \Big)$ is the partition function, again of the the form (2.1), on the resulting lattice $\Lambda^{(m)}$.

The basic idea underlying the development in [1] is very simple. For the choice: $\zeta = b^{d-2}$, $r = 1 - \epsilon$, $0 \leq \epsilon < 1$ of decimation parameters, the r.h.s. in (2.4) gives an upper bound on the the l.h.s., i.e. on the partition function of the previous step. For other choices, such as $\zeta = 1$, $r = 1$, the r.h.s. is a lower bound on the l.h.s. Introducing an interpolating

parameter α, $0 \leq \alpha \leq 1$, one then defines coefficients $\tilde{c}_j(m,\alpha,r)$ and $\tilde{F}_0(m,\alpha)$ interpolating between the upper and lower bound coefficients at $\alpha = 1$ and $\alpha = 0$, respectively. But since there is nothing unique about any one such interpolation, it is expedient to consider more generally a family of such smooth interpolations parametrized by a parameter t in some interval (t_1, t_2). It suffices here to introduce t in $\tilde{F}_0(m,\alpha,t)$ (cf. [1] for details).

Then the upper-lower bounds statement implies that, for each value of t, and any other parameters such as r, there exist some value of the interpolating parameter $0 < \alpha = \alpha_\Lambda^{(m)}(t,r) < 1$, such that

$$\tilde{F}_0(m,\alpha,t)^{|\Lambda^{(m)}|}\, Z_{\Lambda^{(m)}}\Big(\{\tilde{c}_j(m,\alpha,r)\}\Big) = Z_{\Lambda^{(m-1)}} \,. \tag{2.5}$$

Note that, by construction, there is parametrization invariance under shift in t in the l.h.s. of (2.5); in other words, $\alpha_\Lambda^{(m)}(t,r)$ is the level surface fixed by (2.5). This allows determination of the derivatives of $\alpha_\Lambda^{(m)}(t,r)$ w.r.t. t and r. A considerable part of [1] is devoted to various bounds and estimates concerning the behavior of these derivatives.

Iterating this procedure starting from the original lattice, one gets an exact integral representation of the partition function on successively decimated lattices:

$$Z_\Lambda(\beta) = \Big[\prod_{m=1}^{n} \tilde{F}_0(m, \alpha_\Lambda^{(m)}(t_m), t_m)^{|\Lambda|/b^{dm}}\Big]\, Z_{\Lambda^{(n)}}\Big(\{\tilde{c}_j(n, \alpha_\Lambda^{(n)}(t_n))\}\Big)\,. \tag{2.6}$$

To construct something that can serve as a long-distance order parameter one considers the partition function $Z_\Lambda^{(-)}$ in the presence of a vortex, which is introduced by shifting the action on each element of a coclosed plaquette set $\mathcal{V}$ by a non-trivial element ($\tau = -1$) of the group center. To have reflection positivity in all planes in the presence of the flux one may simply replace $Z_\Lambda^{(-)}$ by $Z_\Lambda^{+} \equiv \frac{1}{2}\Big(Z_\Lambda + Z_\Lambda^{(-)}\Big)$. The above development can then be carried through also for Z_Λ^{+} applying the same decimations (2.2) - (2.3). One thus obtains the corresponding integral representation on successively decimated lattices:

$$\begin{aligned} Z_\Lambda^{+} \;&=\; \Big[\prod_{m=1}^{n} \tilde{F}_0(m, \alpha_\Lambda^{+(m)}(t_m), t_m)^{|\Lambda|/b^{dm}}\Big] \\ &\quad\cdot \frac{1}{2}\Big[\, Z_{\Lambda^{(n)}}\Big(\{\tilde{c}_j(n, \alpha_\Lambda^{(+)}(t_n))\}\Big) + Z_{\Lambda^{(n)}}^{(-)}\Big(\{\tilde{c}_j(n, \alpha_\Lambda^{+(n)}(t_n))\}\Big)\Big]\,. \end{aligned} \tag{2.7}$$

As can be seen from (2.7), the external flux presence does not affect the bulk free-energy contributions that resulted from successive blockings. Also, as indicated by the notation, the values $\alpha_\Lambda^{+(m)}(t)$ fixed at each step in this representation are a priori distinct from the values $\alpha_\Lambda^{(m)}(t)$ in the representation (2.6) for $Z_\Lambda(\beta)$.

The vortex free energy order parameter is defined as the ratio of $Z_\Lambda^{(-)}/Z_\Lambda$ or equivalently Z_Λ^{+}/Z_Λ. One may now represent this ratio on successively decimated lattices by inserting our representations (2.6), (2.7) in the numerator and denominator in Z_Λ^{+}/Z_Λ. To account for any small discrepancies between $\alpha_\Lambda^{+(m)}(t)$ and $\alpha_\Lambda^{(m)}(t)$, and arrive at the statement V.1 in [1], the following procedure is followed.

One first utilizes the independent invariance under t-parametrization shifts in numerator and denominator to arrange for cancellation of the bulk free energy pieces due to the

$\tilde{F}_0$-coefficients (trivial characters) between numerator and denominator generated at each successive decimation step. Thus after one decimation:

$$
\begin{aligned}
\frac{Z_\Lambda + Z_\Lambda^{(-)}}{Z_\Lambda} &= \frac{\tilde{F}_0(1,\alpha_\Lambda^{+(1)}(t_1),t_1)^{|\Lambda^{(1)}|}}{\tilde{F}_0(1,\alpha_\Lambda^{(1)}(t_1),t_1)^{|\Lambda^{(1)}|}} \\
&\quad \cdot \frac{\Big[Z_{\Lambda^{(1)}}\Big(\{\tilde{c}_j(1,\alpha_\Lambda^{(+)}(t_1))\}\Big) + Z_{\Lambda^{(1)}}^{(-)}\Big(\{\tilde{c}_j(1,\alpha_\Lambda^{+(1)}(t_1))\}\Big)\Big]}{Z_{\Lambda^{(1)}}\Big(\{\tilde{c}_j(1,\alpha_\Lambda^{(+)}(t_1))\}\Big)} \\
&= \frac{\Big[Z_{\Lambda^{(1)}}\Big(\{\tilde{c}_j(1,\alpha_\Lambda^{(+)}(t_1))\}\Big) + Z_{\Lambda^{(1)}}^{(-)}\Big(\{\tilde{c}_j(1,\alpha_\Lambda^{+(1)}(t_1))\}\Big)\Big]}{Z_{\Lambda^{(1)}}\Big(\{\tilde{c}_j(1,\alpha_\Lambda^{(+)}(t_1'))\}\Big)}
\end{aligned}
\tag{2.8}
$$

by starting with a common $t_1 = t_1^+$ in numerator and denominator, and shifting $t_1 \to t_1'$ in the denominator so that $\tilde{F}_0(1,\alpha_\Lambda^{(1)}(t_1'),t_1')^{|\Lambda^{(1)}|} = \tilde{F}_0(1,\alpha_\Lambda^{+(1)}(t_1),t_1)^{|\Lambda^{(1)}|}$. This is always possible as long as the derivatives of $\alpha_\Lambda^{(m)}(t,r)$ and $\alpha_\Lambda^{+(m)}(t,r)$ w.r.t. t do not vanish, or more precisely are bounded from below by a non-zero constant independent of the lattice volume. As discussed in detail in [1], this can always be ensured by taking the decimation parameter $r < 1$.

Carrying out n successive decimation steps in this manner, one ends up with

$$
\begin{aligned}
\frac{Z_\Lambda + Z_\Lambda^{(-)}}{Z_\Lambda} &= \frac{\tilde{F}_0(n,\alpha_\Lambda^{+(n)}(t_n^+),t_n^+)^{|\Lambda^{(n)}|}}{\tilde{F}_0(n,\alpha_\Lambda^{(n)}(t_n),t_n)^{|\Lambda^{(n)}|}} \\
&\quad \cdot \frac{\Big[Z_{\Lambda^{(n)}}\Big(\{\tilde{c}_j(n,\alpha_\Lambda^{+(n)}(t_n^+))\}\Big) + Z_{\Lambda^{(n)}}^{(-)}\Big(\{\tilde{c}_j(n,\alpha_\Lambda^{+(n)}(t_n^+))\}\Big)\Big]}{Z_{\Lambda^{(n)}}\Big(\{\tilde{c}_j(n,\alpha_\Lambda^{(n)}(t_n))\}\Big)} .
\end{aligned}
\tag{2.9}
$$

Take n such that the flow under the sucessive decimations has entered the strong coupling regime, i.e. let n be *sufficiently large*. At this point one is ready to cancel not only the bulk contributions from the $\tilde{F}_0$ coefficients but also those in the second factor in (2.9), i.e. from all scales. This can be done if one can find a value $t_n = t_n^+ = t_\Lambda^*$ such that $\alpha_\Lambda^{(n)}(t^*) = \alpha_\Lambda^{+(n)}(t^*) \equiv \alpha_\Lambda^{*(n)}$. As shown in [1], and also summarised in [3], such a t_Λ^* exists provided the inequality (5.15) in [1] (restated as (3.5) in [3]) holds. This is a comparison inequality between the derivative w.r.t. the interpolation parameter α of the log of the partition function with versus that without the vortex flux. It is an easy matter to demonstrate that indeed it holds within the strong coupling regime. The result V.1 in [1] (which is incorrectly restated as 'alleged theorem' or claim 2.1 with $n = 1$ in [3], p. 5) then follows, as well as all the consequences spelled out in detail in [1].

3 The Ito-Seiler miss of the non-missing link

As we just saw inequality (5.15) in [1] (restated as (3.5) in [3]) is invoked only after the decimation flow has run into the region of convergence of the strong coupling expansion.

Ito and Seiler make the following curious statement:

"... inequality (3.5), which is, as far as we can see, not proven, even though the author remarks at the beginning of p. 28 of [1]:

"Assume now that under successive decimations the coefficients $c_j^U(m)$ evolve within the convergence radius .... Taking then n sufficiently large, we need establish inequality (5.15) (namely $A \geq A^+$) only at strong coupling." "

These statements in [1] quoted by them were, of course, made in the context of laying out the procedure, outlined in the last part of the previous section, that results in the inequality being invoked only at strong coupling. But, despite the explicit quotations, Ito and Seiler proceed to completely ignore the development that led to them. Instead, they simply state (their italics):

"*It is not clear where and how his claim*, (i.e. the inequality in question), *is proven for large β where the high-temperature expansion never works!* " It is worth noting here that they never indicate the point in [1] at which the inequality is needed at large β. There is, of course, no such point.

To reiterate, *there was never any issue of proving the inequality at large β; the argument in [1] was devised so that the inequality need not be invoked at large β*. It is, in fact, not a priori clear whether it holds at large β. At strong coupling, on the other hand, where it *is* invoked, the proof is immediate, as Ito and Seiler also state.

This misrepresentation of the argument in [1] by Ito and Seiler appears to originate in their incorrect assertions concerning the RG decimations of the potential-moving type that are employed in [1]. On p. 6 (Remark 2.2 (2)) they state:

"The parameter r increases the dimension D from $D = 4$ to $D \geq 4$ from the point of view of the renormalization groups. So we set $r = 1$ in this paper. The introduction of r does not change our argument."

From the point of view of which 'renormalization groups'? We are talking here about precisely specified decimation prescriptions (section 2 in [1]) which are then used to derive various statements. In terms of these, the quoted statement is manifestly incorrect, and in fact absurd. It leads to their outright distortion of the chain of argument in [1].

In implementing the plaquette moving decimation operations that allow various bounds to be derived, and maintaining reflection positivity (positivity of the character expansion coefficients), spacetime dimensionality enters through the actual spacetime lattice on which these operations are carried out, and can only be integer, and through the integer-valued parameter ζ (cf [1]). The statement that changes in the spacetime dimensionality can be absorbed in (small) r-shifts, which renormalize 2-dimensional integrations in hypercell boundaries after the plaquette-moving operations, is non-sensical within the framework of the decimation operations defined in [1].[2]

r is in fact another decimation parameter, one among several that can be introduced in the class of decimations of 'the potential-moving' type, which includes the Migdal-Kadanoff

[2]It is incorrect also outside this framework, i.e. if one considers non-integer dimensions by continuing the decimation transformation formulas – which cannot be interpreted in terms of plaquette moving operations that form the basis of the bounds in [1]. Then both positive and negative expansion coefficients occur, and letting r different from unity certainly cannot be said to amount to raising the spacetime dimension even in a vague qualitative sense, let alone as an algebraically correct statement.

ones as a special case. Allowing variation of this parameter is an integral part of the argument in [1] since such variation ensures that derivatives of the level-surfaces of $\alpha_\Lambda^{(n)}(t,r)$, $\alpha_\Lambda^{+(n)}(t,r)$ do not vanish as mentioned above. Without this assurance, the argument cannot go through. As pointed out in [1], another important outcome of this is to determine on which side of the MK choice of parameters the upper bound parameter values in [1] must lie.

Despite casting everything in the format of definitions, theorems etc, in the usual pseudo-mathematical pretense at 'rigor', general imprecision and confusion characterizes [3] as manifested by the above quoted statements.[3] The authors do not argue from detailed consideration of the complete argument in [1], parts of which they simply ignore; or from any knowledge gained by the actual computation of the decimation flows for a range of decimation parameters, spacetime dimensionalities and gauge groups. They in effect argue by a priori conviction, as well exemplified by their remarks following what they label claim 2.1 (p. 5-6), their misguided discussion section (p. 9), and their reminding us (their reference [2]) that at least one of them has long been advocating the absence of asymptotic freedom and its consequences in non-abelian gauge theory.

References

[1] E.T. Tomboulis, arXiv:0707.2179[hep-th].

[2] E.T. Tomboulis, PoS(LATTICE 2007) 336, arXiv:0710.1894[hep-lat].

[3] K.R. Ito and E. Seiler, arXiv:0707.2179[hep-th].

[3] Various other odd comments appearing in [3], such as the one on positivity of the character expansion coefficients and reflection positivity (p.5, remark 2.1 (1)), are of no overall importance to warrant comment.